Math 210C Lecture 19 Notes

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May 15, 2019

1 Tor Functors and Flat Modules

1.1 Tor of $\mathbb{Z}/n\mathbb{Z}$

Let R be a ring and N be a right R-module. We had a functor $t_N : R\text{-Mod} \to \text{Ab}$ given by $t_N = N \otimes_R \cdot$. This is a right exact functor, so it has a sequence of left derived functors

$$\mathrm{Tor}_i^R(N, \cdot) = L_i t_N.$$

If we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^R(N, C) \longrightarrow N \otimes_R A \longrightarrow N \otimes_R B \longrightarrow N \otimes_R C \longrightarrow 0$$

Example 1.1. Let $R = \mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$. Then we have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ n\ } \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

We get the long exact sequence

$$\cdots \longrightarrow 0 \longrightarrow \underbrace{\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})}_{\cong \mathbb{Z}/n\mathbb{Z}} \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z}/n\mathbb{Z} \xrightarrow{\ \mathrm{id}\ } \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

1.2 Flat modules

Definition 1.1. A right R-module N is **flat** if t_N is exact.

In general, if F is left exact and P is projective, $L_i F(P) = 0$ for $i \geq i$. This motivates the following proposition.

Proposition 1.1. *Projective R^{op}-modules are R^{op}-flat.*

Proof. If P is a projective R^{op}-module, then there exists a projective A such that $P \oplus Q = F$, where F is free. Then if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, so is

$$0 \longrightarrow F \otimes_R A \longrightarrow F \otimes_R B \longrightarrow F \otimes_R C \longrightarrow 0$$

as direct sums commute with the tensor product. We now have

$$0 \longrightarrow (P \otimes A) \oplus (Q \otimes A) \longrightarrow (P \otimes B) \oplus (Q \otimes B) \longrightarrow (P \otimes C) \oplus (Q \otimes C) \longrightarrow 0$$

So we get that

$$0 \longrightarrow P \otimes_R A \longrightarrow P \otimes_R B \longrightarrow P \otimes_R C \longrightarrow 0$$

is exact. $\square$

Proposition 1.2. *Let R be a PID. Then an R-module is flat iff it is R-torsion free.*

Proof. Suppose M is flat over R. Let $0 \neq r \in R$. Then let $\phi_r : R \to R$ send $x \mapsto rx$. Now the map $\mathrm{id}_M \otimes \phi_r : M \otimes_R R \to M \otimes_R R$ is injective, as M is R-flat. But this is just the same as the map $m \mapsto rm$, as $M \otimes_R R \cong M$. So $rm \neq 0$ for all $M \in M$. This holds for all $r \in R$, so M is R-torsion free.

Let M be R-torsion free. Then $M = \bigcup_{N \subseteq M} N = \varinjlim N$, where the indexing is over all finitely generated submodule. Then $M \otimes_R \cong \varinjlim_N N \otimes_R A$ for all A, as t_N is a left adjoint so it commutes with colimits. So it suffices to sho that t_N is left exact for finitely generated, forsion-free N. Such modules are free, so t_N is left exact. This means that t_M is left exact. $\square$

Remark 1.1. Let N be an S-R-bimodule. Then $t_N : R\text{-Mod} \to S\text{-Mod}$, and $\mathrm{Tor}_i^R(N, \cdot) : R\text{-Mod} \to S\text{-Mod}$. Composition with a forgetful functor gives $\mathrm{Tor}_i^R(N, \cdot) : R\text{-Mod} \to \mathrm{Ab}$, as we have already defined.

Example 1.2. Let $A \in \mathrm{Ab}$. We have

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, C) \longrightarrow A \xrightarrow{n} A/nA \longrightarrow 0$$

We get that $\mathrm{Tor}_i^{\mathbb{Z}}(A, \mathbb{Z}) = 0$ for $i \geq 2$, so we get that

$$\mathrm{Tor}_i(A, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} A/nA & i = 0 \\ A[n] & i = 1 \\ 0 & i \geq 2, \end{cases}$$

where $A[n]$ is n-torsion.

Lemma 1.1. *For a right R-module A, the following are equivalent.*

1. *A is flat.*
2. $\operatorname{Tor}_1^R(A, \cdot) = 0$.
3. *$Tor_i^R(A, \cdot) = 0$ for all $i \geq i$.*

Proof. (2) $\implies$ (1): If we have

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0$$

then we get

$$\underbrace{\operatorname{Tor}_1^R(A, B_3)}_{=0} \longrightarrow A \otimes_R B_1 \longrightarrow A \otimes_R B_2 \longrightarrow A \otimes_R B_3 \longrightarrow 0$$

(1) $\implies$ (3): Let $Q_. \to B$ be an R-projective resolution, then we have

$$\cdots \longrightarrow A \otimes_R Q_2 \longrightarrow A \otimes_R Q_1 \longrightarrow A \otimes_R Q_0 \xrightarrow{\mathrm{id} \otimes \varepsilon_B} A \otimes_R B \longrightarrow 0$$

Since t_A is exact, we get that $\operatorname{Tor}_i^R(A, B) = L_i t_A(B) = 0$ for all $i \geq i$. $\square$

1.3 Relatinship between Tor and homology

Proposition 1.3. *Let A be an R^{op}-module, let B be an R-module , and let $P_. \to A$ be a projective resolution of A. Then $\operatorname{Tor}_1^R(A, B) \cong H_i(P_. \otimes_R B)$ for all $i \geq 0$ (natural in B).*

So $\operatorname{Tor}_i^R(\cdot, B)$ are the left derived functors of $\cdot \otimes_R B$.

Proof. Idea: Let $Q_. \to B$ be aprojective resolution of B. Then $P_. \otimes Q_.$ is a "double complex'.' Now let $T_. = \operatorname{Tot}(P_. \otimes_R Q_.)$, where $T_k = \bigoplus_{i+j=k} P_i \otimes_r Q_j$ with differentials $f_i^A \otimes \mathrm{id}_{Q_j} + (-1)^i \mathrm{id}_{o_i} \otimes d_i^B$. We get maps of complexes (up to sign) $T_. \to P_. \otimes_R B$ and $T_. \to A \otimes_R Q_.$. These maps induce isomorphisms on homology. So we get

$$H_i(P \otimes B) \cong H_i(T) \cong H_i(A \otimes Q)$$

for all i. $\square$

Corollary 1.1. *If R is commutative, then $\operatorname{Tor}_i^R(A, B) \cong \operatorname{Tor}_i^R(B, A)$ for R-modules A and B (natural isomorphisms of bifunctors).*

1.4 F-acyclic objects and examples of Tor

Definition 1.2. If $F : \mathcal{C} \to \mathcal{D}$ is a right exact functor, an object $A \in \mathcal{C}$ is F**-acyclic** if $L_i F(A) = 0$ for all $i \geq 1$.

Example 1.3. If A is projective, A is F-acyclic.

Example 1.4. If B is R-flat if and only if B is t_A-acyclic:

$$L_i t_a(B) = \mathrm{Tor}_i^R(A, B) = L_i t'_B(A) = 0$$

for all $i \geq 1$.

Proposition 1.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a right exact functor, and let $C_\cdot \to A$ be a resolution of A by F-acyclic objects. Then $L_i F(A) \cong H_i(F(C))$ for all $i \geq 0$.*

Lemma 1.2. $\mathrm{Tor}_1^{\mathbb{Z}}(A, \cdot) = 0$ *if and only if A is torsion-free.*

We have already shown that the first condition is iff A is flat.

Proof. $\mathrm{Tor}_i^{|} Z(\cdot, B)$ comutes with $\varinjlim$, so it is enough to show this for finitely generated A. So assume $A = \mathbb{Z}^m \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.n_r\mathbb{Z}$. We have a projective resolution

$$0 \longrightarrow \mathbb{Z}^{m+r} \longrightarrow \mathbb{Z}^{m+r} \longrightarrow A \longrightarrow 0.$$

We have $\otimes_Z B : B^{m+r} \to B^{m+r}$ with $\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \cong (B[n_i] \oplus \cdots \oplus B[n_r] = 0)$. We also have $\mathrm{Tor}_j^{|} Z(A, B) = 0$ for all $j \geq 2$. Then $\mathrm{Tor}_1^{\mathbb{Z}}(A, B) = 0$ for all B if and only if $n_1 = \cdots n_r = 1$, which is when A is torsion-free. $\square$

Example 1.5. Let $R = \mathbb{Q}[x, y]$, and consider $\mathbb{Q}$ as an R-module by $x \cdot a = y \cdot b = 0$. That is, $\mathbb{Q} \cong \mathbb{Q}[x, y]/(x, y)$. We have the free resolution

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}[x,y] \arrow[r] & \mathbb{Q}[x,y]^2 \arrow[r] & \mathbb{Q}[x,y] \arrow[r] & \mathbb{Q} \arrow[r] & 0 \\
 & & (x,y) \arrow[ur] & & &
\end{tikzcd}$$

We have that $\mathrm{Tor}_2^R(\mathbb{Q}, \mathbb{Q}) = \ker(\mathbb{Q} \to \mathbb{Q}^2) \cong \mathbb{Q}$. On the other hand, $\mathrm{Tor}_2^R(\mathbb{Q}, \mathbb{Q}) \cong \mathrm{Tor}((x, y), \mathbb{Q})$, where (x, y) is not R-flat.

$$0 \longrightarrow (x, y) \longrightarrow R \longrightarrow \mathbb{Q} \longrightarrow 0$$